

Algorithms: Dynamic Programming (Optimal Binary Search Trees) and Graphs

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DYNAMIC PROGRAMMING

(An algorithmic paradigm not a way of “programming”)

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Dynamic Programming (DP)

Main idea:

- ▶ Remember calculations already made
- ▶ Saves enormous amounts of computation

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- ▶ Saves enormous amounts of computation

Allows to solve many optimization problems

- ▶ Always at least one question in google code jam needs DP

Key elements in designing a DP-algorithm

Optimal substructure

- ▶ Show that a solution to a problem consists of **making a choice**, which leaves one or several subproblems to solve

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Overlapping subproblems

- ▶ A naive recursive algorithm may revisit the same (sub)problem over and over.
- ▶ **Top-down with memoization**
Solve recursively but store each result in a table
- ▶ **Bottom-up**
Sort the subproblems and solve the smaller ones first; that way, when solving a subproblem, have already solved the smaller subproblems we need



ROD CUTTING

Rod cutting

Definition

INPUT: A length n and table of prices p_i , for $i = 1, \dots, n$

OUTPUT: The maximum revenue obtainable for rods whose lengths sum up to n , computed as the sum of the prices for the individual rods.

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(a)



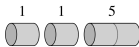
(b)



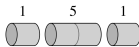
(c)



(d)



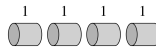
(e)



(f)



(g)



(h)

Dynamic programming algorithm

Choice:

Dynamic programming algorithm

Choice: where to make the leftmost cut

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Hence, if we let $r(n)$ be the optimal revenue from a rod of length n , we can **express $r(n)$ recursively** as follows

$$r(n) = \begin{cases} 0 & \text{if } n = 0 \\ \max_{1 \leq i \leq n} \{p_i + r(n - i)\} & \text{otherwise if } n \geq 1 \end{cases}$$

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Optimal substructure: Solve recurrence using top-down with memoization or bottom-up which yields an algorithm that runs in time $\Theta(n^2)$.

Parenthesization	Cost computation	Cost
$A \times ((B \times C) \times D)$	$20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100$	120,200
$(A \times (B \times C)) \times D$	$20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$	60,200
$(A \times B) \times (C \times D)$	$50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$	7,000

MATRIX-CHAIN MULTIPLICATION

Matrix-chain multiplication

Definition

INPUT: A chain $\langle A_1, A_2, \dots, A_n \rangle$ of n matrices, where for $i = 1, 2, \dots, n$, matrix A_i has dimension $p_{i-1} \times p_i$

OUTPUT: A full parenthesization of the product $A_1 A_2 \cdots A_n$ in a way that minimizes the number of scalar multiplications

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OUTPUT: A full parenthesization of the product $A_1 A_2 \cdots A_n$ in a way that minimizes the number of scalar multiplications

Example: Optimal parenthesization of $A_{4,3} \times B_{3,5} \times C_{5,2}$ is

$$(A_{4,3} \times (B_{3,5} \times C_{5,2}))$$

and requires $3 \cdot 5 \cdot 2 + 4 \cdot 3 \cdot 2$ multiplications.

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Choice: where to make the outermost parenthesis

$$(A_1 \cdots A_k)(A_{k+1} \cdots A_n)$$

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Hence, if we let $m[i, j]$ be the optimal value for chain multiplication of matrices A_i, \dots, A_j , we can express $m[i, j]$ recursively as follows

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\} & \text{otherwise if } i < j \end{cases}$$

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Overlapping subproblems: Solve recurrence using top-down with memoization or bottom-up which yields an algorithm that runs in time $\Theta(n^3)$.

LONGEST COMMON SUBSEQUENCE

Longest common subsequence

Definition

INPUT: 2 sequences, $X = \langle x_1, \dots, x_m \rangle$ and $Y = \langle y_1, \dots, y_n \rangle$.

OUTPUT: A subsequence common to both whose length is longest.
A subsequence doesn't have to be consecutive, but it has to be in order

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Example:

heroically
scholarly

The diagram illustrates the longest common subsequence (LCS) between the words "heroically" and "scholarly". Vertical lines connect the matching characters: 'h' to 'h', 'e' to 'e', 'r' to 'r', 'o' to 'o', 'i' to 'i', 'c' to 'c', 'a' to 'a', 'l' to 'l', and 'l' to 'l'. The 'y' at the end of "heroically" does not have a corresponding character in "scholarly".

Dynamic programming comes to the rescue

Start at the end of both words and move to the left step-by-step

BABDBA

DACBCBA

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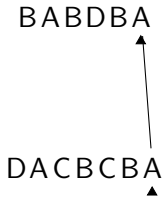
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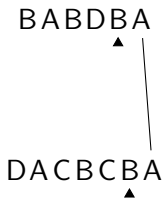
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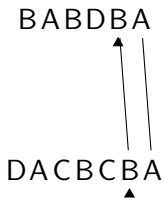
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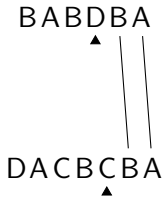
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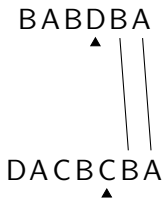


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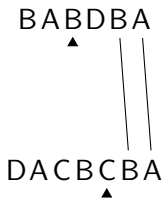


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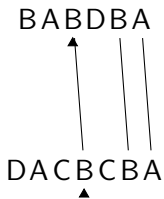


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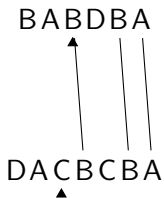


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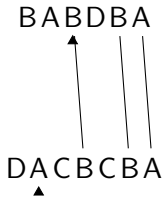


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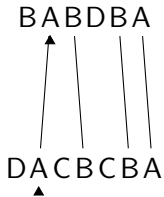


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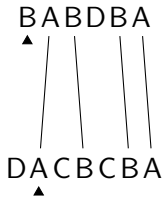


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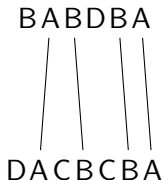


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Let $X_i = \langle x_1, x_2, \dots, x_i \rangle$ and $Y_j = \langle y_1, y_2, \dots, y_j \rangle$

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If $x_i = y_j$ then either

- ▶ OPT “matches” x_i with y_j and remaining OPT is in (X_{i-1}, Y_{j-1}) ;
- ▶ OPT is in (X_{i-1}, Y_j) ; or
- ▶ OPT is in (X_i, Y_{j-1})

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We proved that we can assume that OPT “matches” x_i with y_j if they are equal so we can simplify the first case

Recursive formulation

Define $c[i, j] = \text{length of LCS of } X_i \text{ and } Y_j$. We want $c[m, n]$

$$c[i, j] = \left\{ \begin{array}{l} \end{array} \right.$$

Recursive formulation

Define $c[i, j] = \text{length of LCS of } X_i \text{ and } Y_j$. We want $c[m, n]$

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$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i - 1, j - 1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \end{cases}$$

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- ▶ Naive implementation solves same problems many many times
- ▶ Solve with Bottom-Up or Top-Down with Memoization in time $O(m \cdot n)$.

Pseudocode and analysis

```
LCS-LENGTH( $X, Y, m, n$ )  
  let  $b[1..m, 1..n]$  and  $c[0..m, 0..n]$  be new tables  
  for  $i = 1$  to  $m$   
     $c[i, 0] = 0$   
  for  $j = 0$  to  $n$   
     $c[0, j] = 0$   
  for  $i = 1$  to  $m$   
    for  $j = 1$  to  $n$   
      if  $x_i == y_j$   
         $c[i, j] = c[i - 1, j - 1] + 1$   
         $b[i, j] = \nwarrow$   
      else if  $c[i - 1, j] \geq c[i, j - 1]$   
         $c[i, j] = c[i - 1, j]$   
         $b[i, j] = \uparrow$   
      else  $c[i, j] = c[i, j - 1]$   
         $b[i, j] = \leftarrow$   
  return  $c$  and  $b$ 
```

Pseudocode and analysis

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    for  $j = 1$  to  $n$   
      if  $x_i == y_j$   
         $c[i, j] = c[i - 1, j - 1] + 1$   
         $b[i, j] = \text{"↖"}$   
      else if  $c[i - 1, j] \geq c[i, j - 1]$   
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- ▶ Time dominated by instructions inside the two nested loops which execute $m \cdot n$ times

Pseudocode and analysis

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  return  $c$  and  $b$ 
```

- ▶ Time dominated by instructions inside the two nested loops which execute $m \cdot n$ times
- ▶ Total time is $\Theta(m \cdot n)$.



OPTIMAL BINARY SEARCH TREES

Searching on Facebook



More popular than



Optimal binary search trees

- ▶ Given sequence $K = \langle k_1, k_2, \dots, k_n \rangle$ of n distinct keys, sorted ($k_1 < k_2 < \dots < k_n$).
- ▶ Want to build a binary search tree from the keys

Optimal binary search trees

- ▶ Given sequence $K = \langle k_1, k_2, \dots, k_n \rangle$ of n distinct keys, sorted ($k_1 < k_2 < \dots < k_n$).
- ▶ Want to build a binary search tree from the keys
- ▶ For k_i , have probability p_i that a search is for k_i
- ▶ Want BST with minimum expected search cost

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- ▶ Want to build a binary search tree from the keys
- ▶ For k_i , have probability p_i that a search is for k_i
- ▶ Want BST with minimum expected search cost
- ▶ Actual cost = # of items examined

For key k_i , cost = $\text{depth}_T(k_i) + 1$, where $\text{depth}_T(k_i)$ denotes the depth of k_i in BST T

Optimal binary search trees

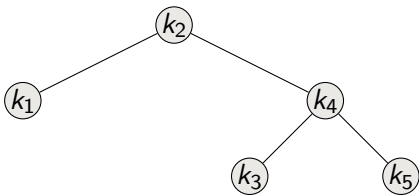
- ▶ Given sequence $K = \langle k_1, k_2, \dots, k_n \rangle$ of n distinct keys, sorted ($k_1 < k_2 < \dots < k_n$).
- ▶ Want to build a binary search tree from the keys
- ▶ For k_i , have probability p_i that a search is for k_i
- ▶ Want BST with minimum expected search cost
- ▶ Actual cost = # of items examined

For key k_i , cost = $\text{depth}_T(k_i) + 1$, where $\text{depth}_T(k_i)$ denotes the depth of k_i in BST T

$$\begin{aligned}\mathbb{E}[\text{search cost in } T] &= \sum_{i=1}^n (\text{depth}_T(k_i) + 1) p_i \\ &= 1 + \sum_{i=1}^n \text{depth}_T(k_i) \cdot p_i\end{aligned}$$

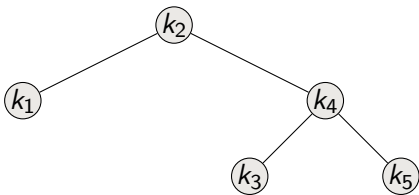
Example

i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3



Example

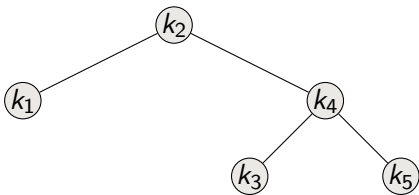
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p_i	.25	.2	.05	.2	.3



i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	.25
2	0	0
3	2	.1
4	1	.2
5	2	.6
		<hr/> 1.15

Example

i	1	2	3	4	5
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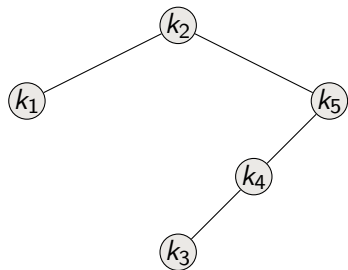


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4	1	.2
5	2	.6
		<hr/> 1.15

Therefore, $\mathbb{E}[\text{search cost}] = 2.15$

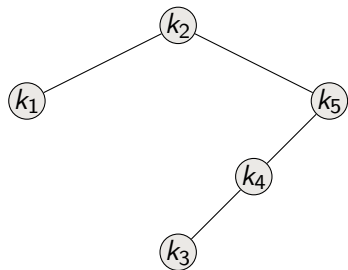
Example

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Example

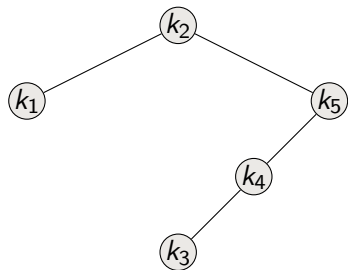
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1	1	.25
2	0	0
3	3	.15
4	2	.4
5	1	.3
		<hr/> 1.10

Example

i	1	2	3	4	5
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1	1	.25
2	0	0
3	3	.15
4	2	.4
5	1	.3
		<hr/> 1.10

Therefore, $\mathbb{E}[\text{search cost}] = 2.10$, which turns out to be optimal

Observations

- ▶ Optimal BST might not have smallest height
- ▶ Optimal BST might not have highest-probability key at root

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DP comes to the rescue :)

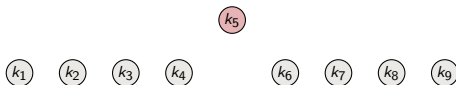
Optimal substructure

A binary search tree can be built by first picking the root and then building the subtrees recursively



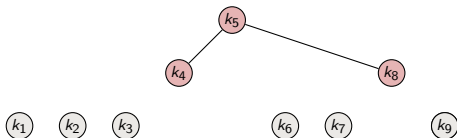
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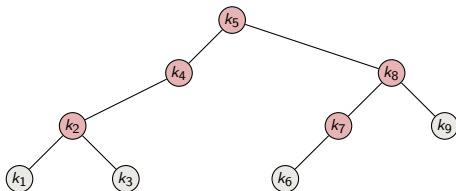
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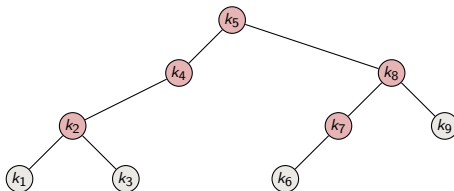
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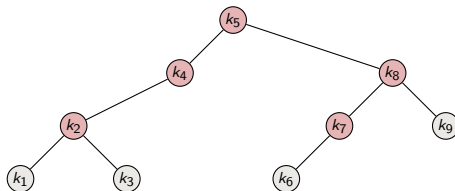
A binary search tree can be built by first picking the root and then building the subtrees recursively



$$\mathbb{E}[\text{search cost}] = p_5 + 2p_4 + 3p_2 + 4p_1 + 4p_3 + 2p_8 + 3p_7 + 3p_9 + 4p_6$$

Optimal substructure

A binary search tree can be built by first picking the root and then building the subtrees recursively



$$\mathbb{E}[\text{search cost}] = p_5$$

$$+ p_1 + p_2 + p_3 + p_4 + \mathbb{E}[\text{search cost left subtree}]$$

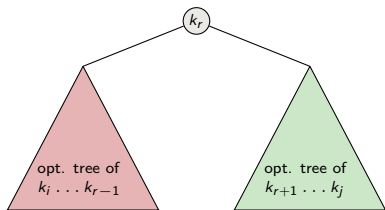
$$+ p_6 + p_7 + p_8 + p_9 + \mathbb{E}[\text{search cost right subtree}]$$

Optimal substructure

A binary search tree can be built by first picking the root and then building the subtrees recursively

After picking root solution to subtrees must be optimal

Build tree of nodes $k_i < k_{i+1} < \dots < k_{j-1} < k_j$ by selecting best root r :



$$\mathbb{E}[\text{search cost}] = p_r$$

$$+ p_i + \dots + p_{r-1} + \mathbb{E}[\text{search cost left subtree}]$$

$$+ p_{r+1} + \dots + p_j + \mathbb{E}[\text{search cost right subtree}]$$

Recursive formulation

- ▶ Let $e[i, j] =$ expected search cost of optimal BST of $k_i \dots k_j$

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$$e[i, j] = \left\{ \right.$$

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$$e[i, j] = \begin{cases} & \text{if } i = j + 1 \end{cases}$$

Recursive formulation

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- ▶ Let $e[i, j]$ = expected search cost of optimal BST of $k_i \dots k_j$

$$e[i, j] = \begin{cases} 0 & \text{if } i = j + 1 \\ \min_{i \leq r \leq j} \{e[i, r - 1] + e[r + 1, j] + \sum_{\ell=i}^j p_{\ell}\} & \text{if } i \leq j \end{cases}$$

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- ▶ Solve using bottom-up or top-down with memoization

Bottom-up example

i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3

$$e[i, j] = \begin{cases} 0 & \text{if } i = j + 1 \\ \min_{i \leq r \leq j} \{ e[i, r-1] + e[r+1, j] + \sum_{\ell=i}^j p_{\ell} \} & \text{if } i \leq j \end{cases}$$

e	0	1	2	3	4	5
1						
2						
3						
4						
5						
6						

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e	0	1	2	3	4	5
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Bottom-up example

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2		0				
3			0			
4				0		
5					0	
6						0

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e	0	1	2	3	4	5
1	0	.25	.65			
2		0	.2	.3		
3			0	.05	.3	
4				0	.2	.7
5					0	.3
6						0

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e	0	1	2	3	4	5
1	0	.25	.65	.8		
2		0	.2	.3		
3			0	.05	.3	
4				0	.2	.7
5					0	.3
6						0

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4				0	.2	.7
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e	0	1	2	3	4	5
1	0	.25	.65	.8		
2		0	.2	.3	.75	
3			0	.05	.3	.85
4				0	.2	.7
5					0	.3
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Bottom-up example

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e	0	1	2	3	4	5
1	0	.25	.65	.8	1.25	
2		0	.2	.3	.75	
3			0	.05	.3	.85
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1	0	.25	.65	.8	1.25	2.1
2		0	.2	.3	.75	1.35
3			0	.05	.3	.85
4				0	.2	.7
5					0	.3
6						0

Optimal BST has expected search cost 2.1

Can save decisions to reconstruct tree

Pseudocode of bottom-up

```
OPTIMAL-BST( $p, q, n$ )
  let  $e[1 \dots n + 1, 0 \dots n]$ ,  $w[1 \dots n + 1, 0 \dots n]$ , and  $root[1 \dots n, 1 \dots n]$  be new tables
  for  $i = 1$  to  $n + 1$ 
     $e[i, i - 1] = 0$ 
     $w[i, i - 1] = 0$ 
  for  $l = 1$  to  $n$ 
    for  $i = 1$  to  $n - l + 1$ 
       $j = i + l - 1$ 
       $e[i, j] = \infty$ 
       $w[i, j] = w[i, j - 1] + p_j$ 
      for  $r = i$  to  $j$ 
         $t = e[i, r - 1] + e[r + 1, j] + w[i, j]$ 
        if  $t < e[i, j]$ 
           $e[i, j] = t$ 
           $root[i, j] = r$ 
  return  $e$  and  $root$ 
```

$e[i, j]$ records the expected search cost of optimal BST of k_i, \dots, k_j

$r[i, j]$ records the best root in optimal BST of k_i, \dots, k_j

$w[i, j]$ records $\sum_{\ell=i}^j p_{\ell}$

Runtime Analysis

OPTIMAL-BST(p, q, n)

let $e[1 \dots n + 1, 0 \dots n]$, $w[1 \dots n + 1, 0 \dots n]$, and $root[1 \dots n, 1 \dots n]$ be new tables

for $i = 1$ **to** $n + 1$

$e[i, i - 1] = 0$

$w[i, i - 1] = 0$

for $l = 1$ **to** n

for $i = 1$ **to** $n - l + 1$

$j = i + l - 1$

$e[i, j] = \infty$

$w[i, j] = w[i, j - 1] + p_j$

for $r = i$ **to** j

$t = e[i, r - 1] + e[r + 1, j] + w[i, j]$

if $t < e[i, j]$

$e[i, j] = t$

$root[i, j] = r$

return e and $root$

Runtime Analysis

```
OPTIMAL-BST( $p, q, n$ )  
  let  $e[1 \dots n + 1, 0 \dots n]$ ,  $w[1 \dots n + 1, 0 \dots n]$ , and  $root[1 \dots n, 1 \dots n]$  be new tables  
  for  $i = 1$  to  $n + 1$   
     $e[i, i - 1] = 0$   
     $w[i, i - 1] = 0$   
  for  $l = 1$  to  $n$   
    for  $i = 1$  to  $n - l + 1$   
       $j = i + l - 1$   
       $e[i, j] = \infty$   
       $w[i, j] = w[i, j - 1] + p_j$   
      for  $r = i$  to  $j$   
         $t = e[i, r - 1] + e[r + 1, j] + w[i, j]$   
        if  $t < e[i, j]$   
           $e[i, j] = t$   
           $root[i, j] = r$   
  return  $e$  and  $root$ 
```

- ▶ Runtime dominated by three nested loops: total time is $\Theta(n^3)$

Runtime Analysis

```
OPTIMAL-BST( $p, q, n$ )  
  let  $e[1 \dots n + 1, 0 \dots n]$ ,  $w[1 \dots n + 1, 0 \dots n]$ , and  $root[1 \dots n, 1 \dots n]$  be new tables  
  for  $i = 1$  to  $n + 1$   
     $e[i, i - 1] = 0$   
     $w[i, i - 1] = 0$   
  for  $l = 1$  to  $n$   
    for  $i = 1$  to  $n - l + 1$   
       $j = i + l - 1$   
       $e[i, j] = \infty$   
       $w[i, j] = w[i, j - 1] + p_j$   
      for  $r = i$  to  $j$   
         $t = e[i, r - 1] + e[r + 1, j] + w[i, j]$   
        if  $t < e[i, j]$   
           $e[i, j] = t$   
           $root[i, j] = r$   
  return  $e$  and  $root$ 
```

- ▶ Runtime dominated by three nested loops: total time is $\Theta(n^3)$
- ▶ Alternatively, $\Theta(n^2)$ cells to fill in

Runtime Analysis

```
OPTIMAL-BST( $p, q, n$ )  
  let  $e[1 \dots n + 1, 0 \dots n]$ ,  $w[1 \dots n + 1, 0 \dots n]$ , and  $root[1 \dots n, 1 \dots n]$  be new tables  
  for  $i = 1$  to  $n + 1$   
     $e[i, i - 1] = 0$   
     $w[i, i - 1] = 0$   
  for  $l = 1$  to  $n$   
    for  $i = 1$  to  $n - l + 1$   
       $j = i + l - 1$   
       $e[i, j] = \infty$   
       $w[i, j] = w[i, j - 1] + p_j$   
      for  $r = i$  to  $j$   
         $t = e[i, r - 1] + e[r + 1, j] + w[i, j]$   
        if  $t < e[i, j]$   
           $e[i, j] = t$   
           $root[i, j] = r$   
  return  $e$  and  $root$ 
```

- ▶ Runtime dominated by three nested loops: total time is $\Theta(n^3)$
- ▶ Alternatively, $\Theta(n^2)$ cells to fill in
Most cells take $\Theta(n)$ time to fill in

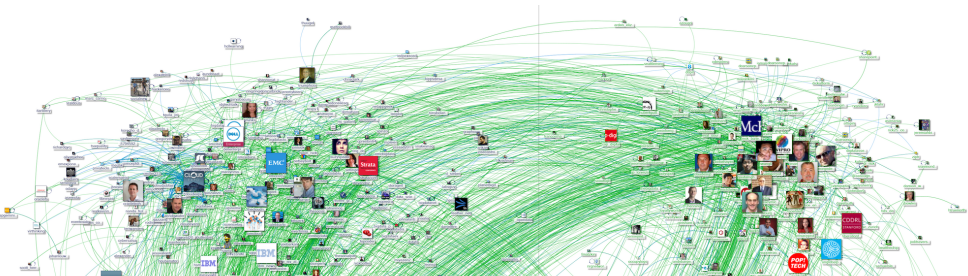
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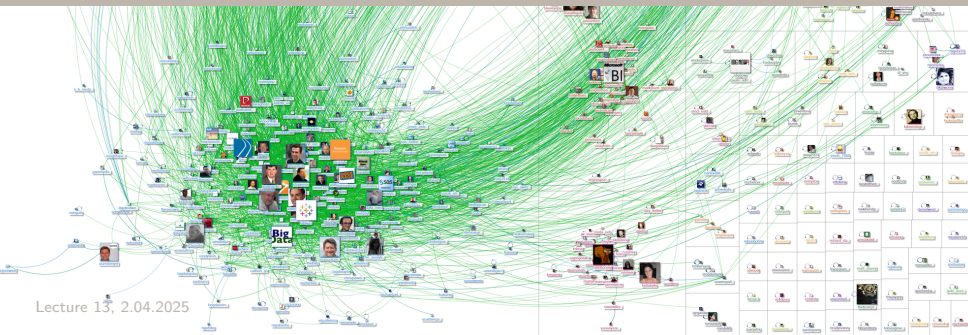
- ▶ Runtime dominated by three nested loops: total time is $\Theta(n^3)$
- ▶ Alternatively, $\Theta(n^2)$ cells to fill in
Most cells take $\Theta(n)$ time to fill in
Hence, total time is $\Theta(n^3)$

Summary of Dynamic Programming

- ▶ Identify choices and optimal substructure
- ▶ Write optimal solution recursively as a function of smaller subproblems
- ▶ Use top-down with memoization or bottom-up to solve the recursion efficiently (without repeatedly solving the same subproblems)
- ▶ Do a lot of exercises!



GRAPHS



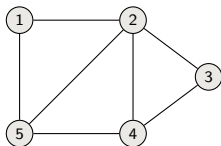
Graphs

A graph $G = (V, E)$ consists of

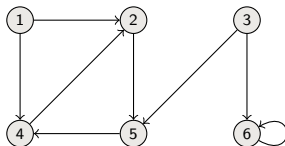
- ▶ a vertex set V
- ▶ an edge set E that contain (ordered) pairs of vertices

A graph can be undirected, directed, vertex-weighted, edge-weighted, etc.

Undirected Graph



Directed Graph

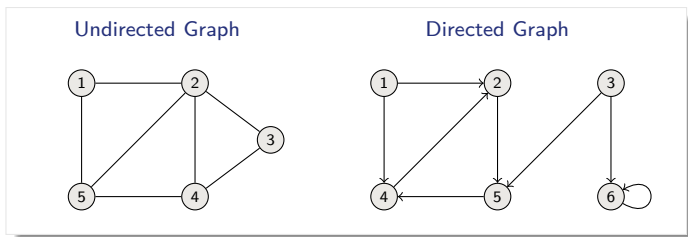


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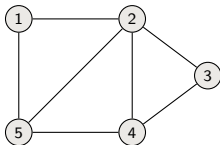


How to represent a graph in the computer?

Adjacency Lists

- ▶ Array Adj of $|V|$ lists, one per vertex

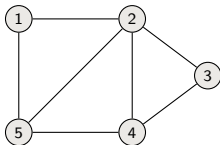
Undirected Graph



Adjacency Lists

- ▶ Array Adj of $|V|$ lists, one per vertex
- ▶ Vertex u 's list has all vertices v such that $(u, v) \in E$ (works for both undirected and directed graphs)

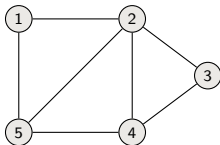
Undirected Graph



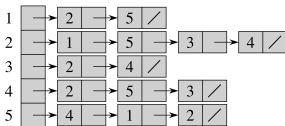
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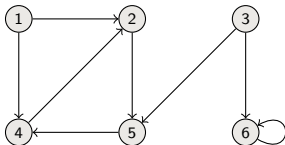
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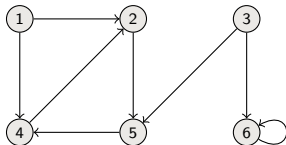
Directed Graph



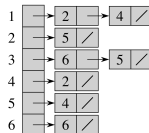
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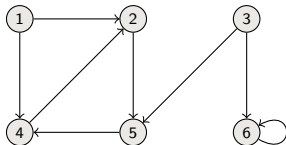
Adjacency list Adj



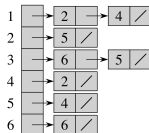
Adjacency Lists

- ▶ Array Adj of $|V|$ lists, one per vertex
- ▶ Vertex u 's list has all vertices v such that $(u, v) \in E$ (works for both undirected and directed graphs)
- ▶ In pseudocode, we will denote the array as attribute $G.Adj$, so we will see notation such as $G.Adj[u]$.

Directed Graph



Adjacency list Adj

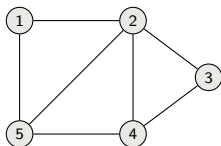


Adjacency matrix

- A $|V| \times |V|$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Undirected Graph

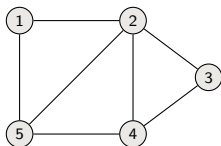


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Undirected Graph



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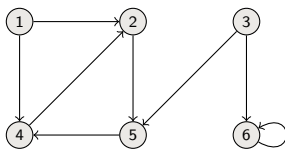
	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

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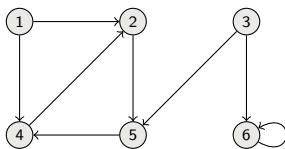


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Directed Graph



Adjacency matrix

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Comparison of adjacency list and adjacency matrix

Adjacency list

Adjacency matrix

Space

Space

Comparison of adjacency list and adjacency matrix

Adjacency list

$$\text{Space} = \Theta(V + E)$$

Adjacency matrix

Space

Comparison of adjacency list and adjacency matrix

Adjacency list

$$\text{Space} = \Theta(V + E)$$

Adjacency matrix

$$\text{Space} = \Theta(V^2)$$

Comparison of adjacency list and adjacency matrix

Adjacency list

Space = $\Theta(V + E)$

Time: to list all vertices adjacent to u :

Adjacency matrix

Space = $\Theta(V^2)$

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Comparison of adjacency list and adjacency matrix

Adjacency list

Space = $\Theta(V + E)$

Time: to list all vertices adjacent to u : $\Theta(\text{degree}(u))$

Adjacency matrix

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Comparison of adjacency list and adjacency matrix

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Comparison of adjacency list and adjacency matrix

Adjacency list

Space = $\Theta(V + E)$

Time: to list all vertices adjacent to u : $\Theta(\text{degree}(u))$

Time: to determine whether $(u, v) \in E$:

Adjacency matrix

Space = $\Theta(V^2)$

Time: to list all vertices adjacent to u : $\Theta(V)$

Time: to determine whether $(u, v) \in E$:

Comparison of adjacency list and adjacency matrix

Adjacency list

Space = $\Theta(V + E)$

Time: to list all vertices adjacent to u : $\Theta(\text{degree}(u))$

Time: to determine whether $(u, v) \in E$: $O(\text{degree}(u))$

Adjacency matrix

Space = $\Theta(V^2)$

Time: to list all vertices adjacent to u : $\Theta(V)$

Time: to determine whether $(u, v) \in E$:

Comparison of adjacency list and adjacency matrix

Adjacency list

Space = $\Theta(V + E)$

Time: to list all vertices adjacent to u : $\Theta(\text{degree}(u))$

Time: to determine whether $(u, v) \in E$: $O(\text{degree}(u))$

Adjacency matrix

Space = $\Theta(V^2)$

Time: to list all vertices adjacent to u : $\Theta(V)$

Time: to determine whether $(u, v) \in E$: $\Theta(1)$

Comparison of adjacency list and adjacency matrix

Adjacency list

Space = $\Theta(V + E)$

Time: to list all vertices adjacent to u : $\Theta(\text{degree}(u))$

Time: to determine whether $(u, v) \in E$: $O(\text{degree}(u))$

Adjacency matrix

Space = $\Theta(V^2)$

Time: to list all vertices adjacent to u : $\Theta(V)$

Time: to determine whether $(u, v) \in E$: $\Theta(1)$

We can extend both representations to include other attributes such as edge weights

TRAVERSING/SEARCHING A GRAPH

Breadth-First Search

Definition

INPUT: Graph $G = (V, E)$, either directed or undirected and source vertex $s \in V$

OUTPUT: $v.d = \text{distance (smallest number of edges) from } s \text{ to } v$,
for all $v \in V$

Breadth-First Search

Definition

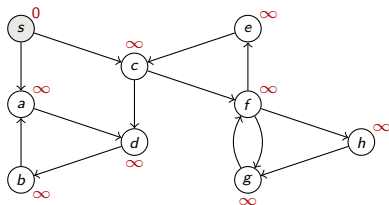
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Idea:

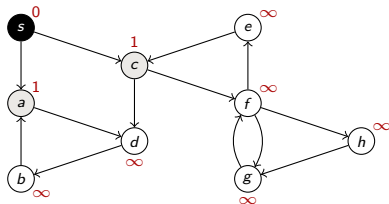
- ▶ Send a wave out from s
- ▶ First hits all vertices 1 edge from s
- ▶ From there, hits all vertices 2 edges from s ...

Example of Breadth-first search



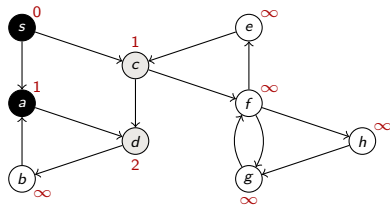
Queue $Q = s$

Example of Breadth-first search



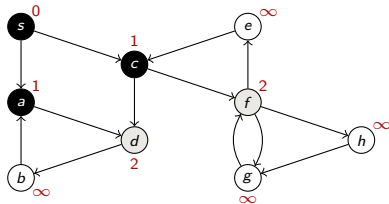
Queue $Q = a, c$

Example of Breadth-first search



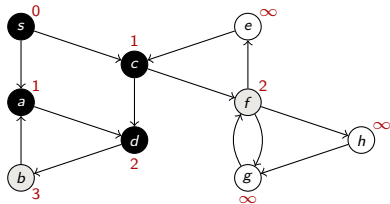
Queue $Q = c, d$

Example of Breadth-first search



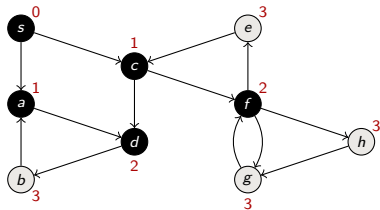
Queue $Q = d, f$

Example of Breadth-first search



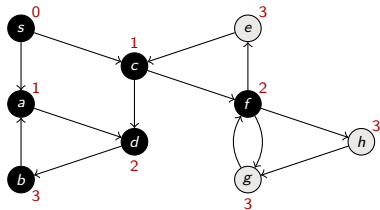
Queue $Q = f, b$

Example of Breadth-first search



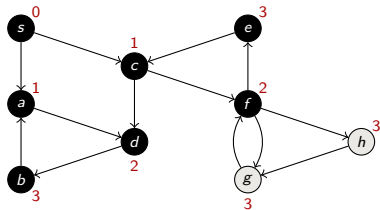
Queue Q = b,e,g,h

Example of Breadth-first search



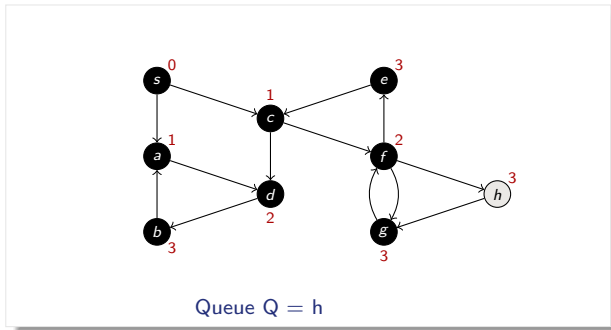
Queue Q = e,g,h

Example of Breadth-first search

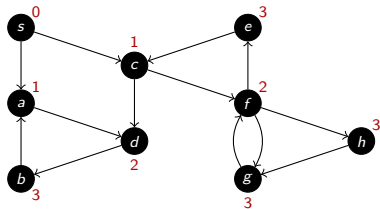


Queue $Q = g, h$

Example of Breadth-first search



Example of Breadth-first search



Queue $Q = \text{nil}$

Pseudocode of Breadth-first search

BFS(V, E, s)

for each $u \in V - \{s\}$

$u.d = \infty$

$s.d = 0$

$Q = \emptyset$

 ENQUEUE(Q, s)

while $Q \neq \emptyset$

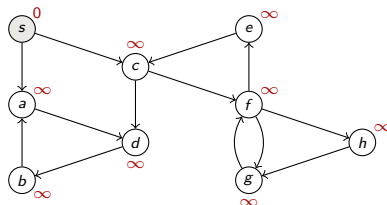
$u = \text{DEQUEUE}(Q)$

for each $v \in G.\text{Adj}[u]$

if $v.d == \infty$

$v.d = u.d + 1$

 ENQUEUE(Q, v)



Queue $Q = s$

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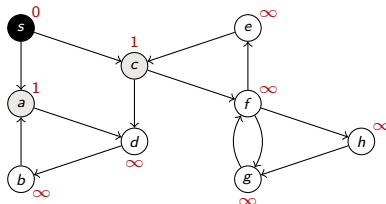
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Queue $Q = a, c$

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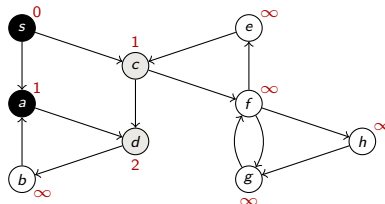
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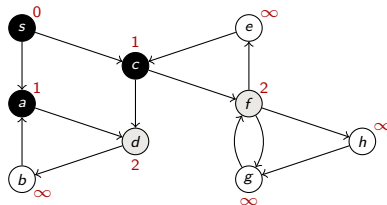
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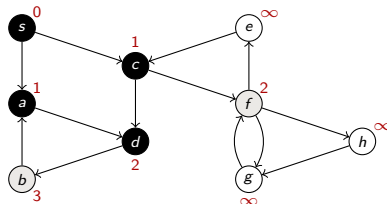
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Queue $Q = f, b$

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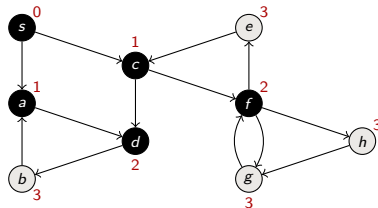
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Queue $Q = b, e, g, h$

Pseudocode of Breadth-first search

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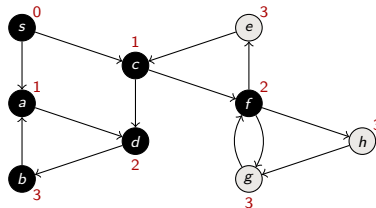
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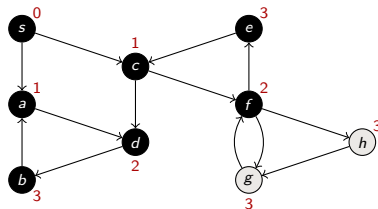
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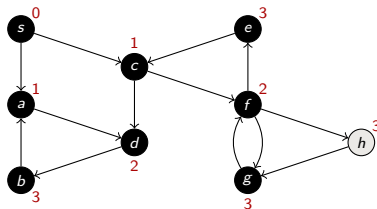
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 ENQUEUE(Q, v)



Queue $Q = h$

Pseudocode of Breadth-first search

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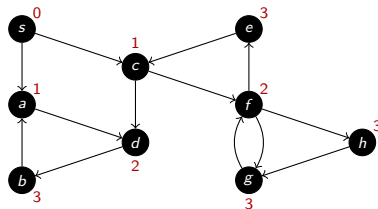
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Informal Idea of correctness (formal proof in book):

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- ▶ Suppose that $v.d$ is greater than the shortest distance from s to v
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Runtime analysis:

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Runtime analysis: $O(V+E)$

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- ▶ Suppose that $v.d$ is greater than the shortest distance from s to v
- ▶ but since algorithm repeatedly considers the vertices closest to the root (by adding them to the queue) this cannot happen

Runtime analysis: $O(V+E)$

- ▶ $O(V)$ because each vertex enqueued at most once

Analysis

Informal Idea of correctness (formal proof in book):

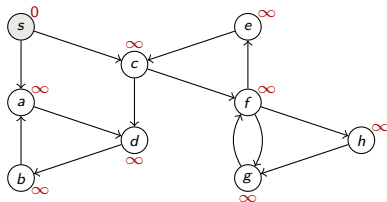
- ▶ Suppose that $v.d$ is greater than the shortest distance from s to v
- ▶ but since algorithm repeatedly considers the vertices closest to the root (by adding them to the queue) this cannot happen

Runtime analysis: $O(V+E)$

- ▶ $O(V)$ because each vertex enqueued at most once
- ▶ $O(E)$ because every vertex dequeued at most once and we examine (u, v) only when u is dequeued. Therefore, every edge examined at most once if directed and at most twice if undirected

Final notes on BFS

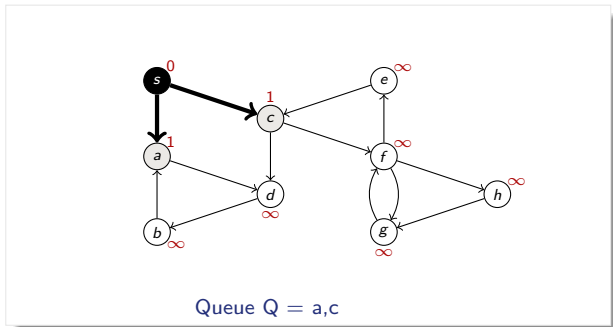
- ▶ BFS may not reach all the vertices
- ▶ We can save the shortest path tree by keeping track of the edge that discovered the vertex



Queue $Q = s$

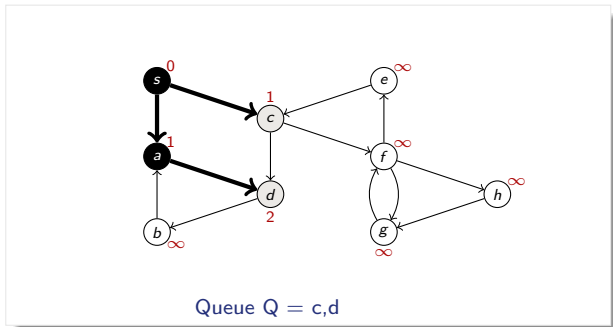
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- ▶ BFS may not reach all the vertices
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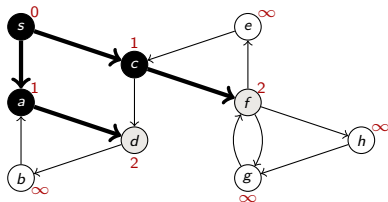
Final notes on BFS

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Final notes on BFS

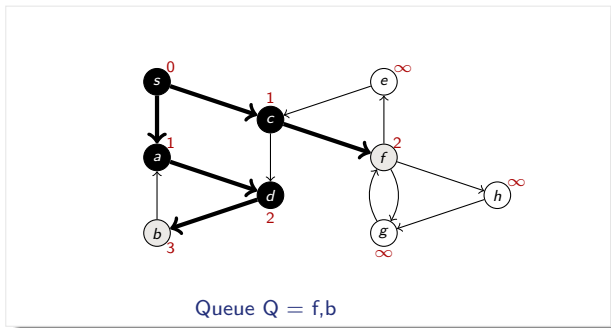
- ▶ BFS may not reach all the vertices
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Queue $Q = d, f$

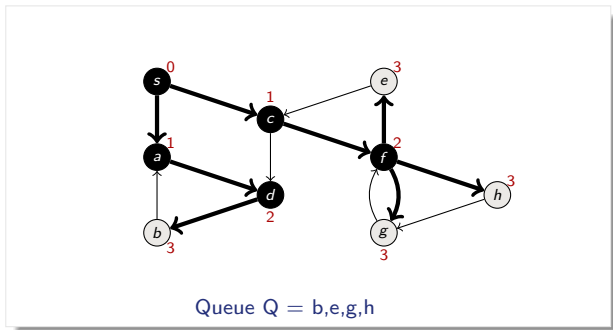
Final notes on BFS

- ▶ BFS may not reach all the vertices
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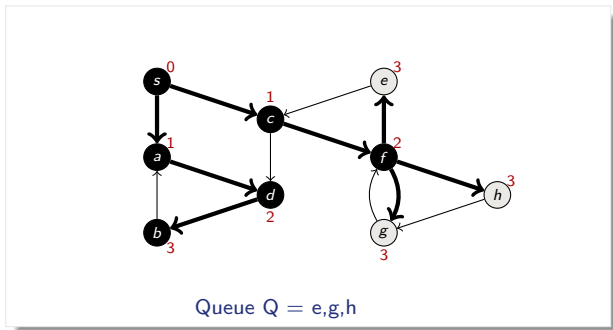
Final notes on BFS

- ▶ BFS may not reach all the vertices
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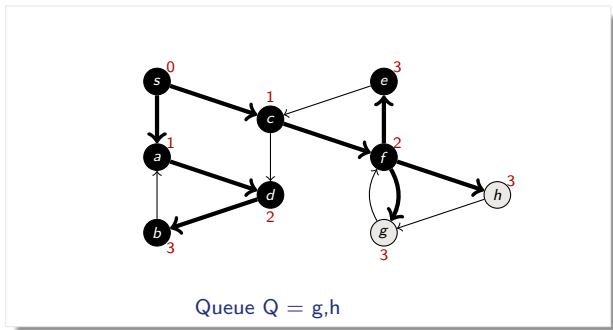
Final notes on BFS

- ▶ BFS may not reach all the vertices
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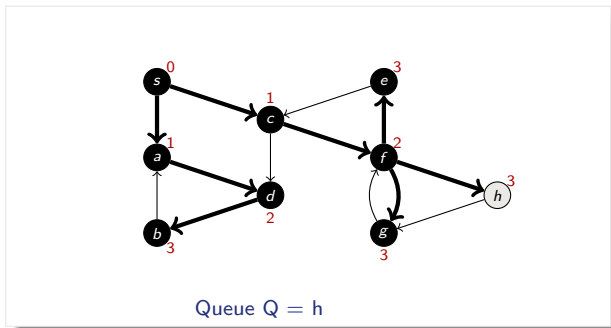
Final notes on BFS

- ▶ BFS may not reach all the vertices
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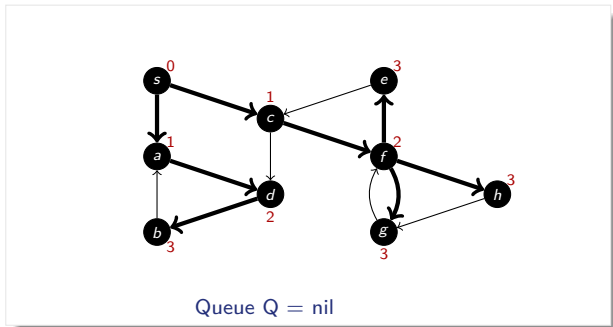
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Final notes on BFS

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Depth-First Search

Definition

INPUT: Graph $G = (V, E)$, either directed or undirected

OUTPUT: 2 timestamps on each vertex: $v.d = \text{discovery time}$ and $v.f = \text{finishing time}$

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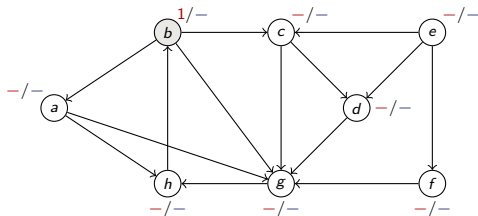
Idea:

- ▶ Methodically explore *every* edge
- ▶ Start over from different vertices as necessary
- ▶ As soon as we discover a vertex explore from it,
 - ▶ Unlike BFS, which explores vertices that are close to a source first

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)

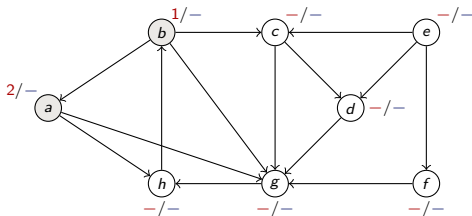


time = 1

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
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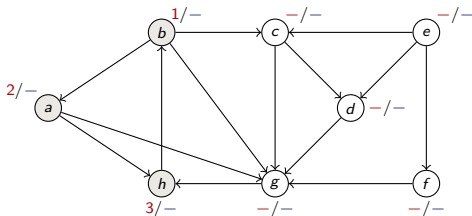


time = 2

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)

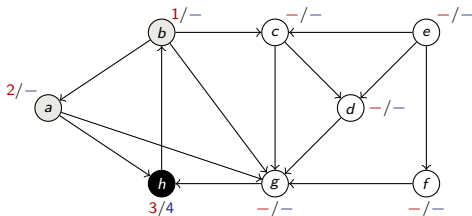


time = 3

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
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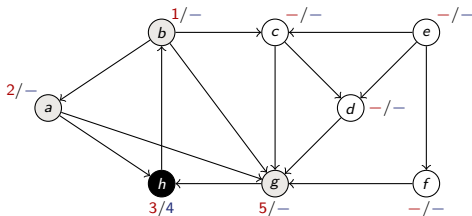


time = 4

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)

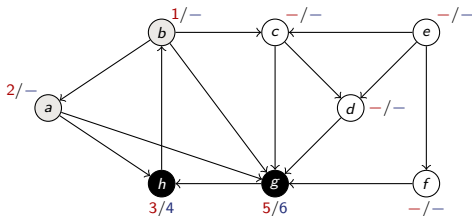


time = 5

Example of DFS

As DFS progresses, every vertex has a color:

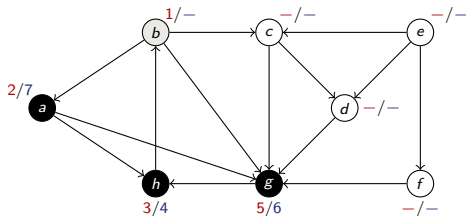
- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
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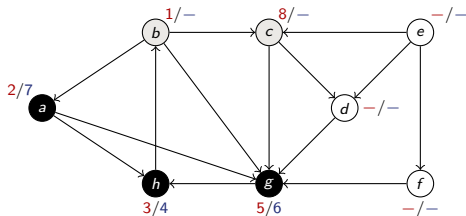


time = 7

Example of DFS

As DFS progresses, every vertex has a color:

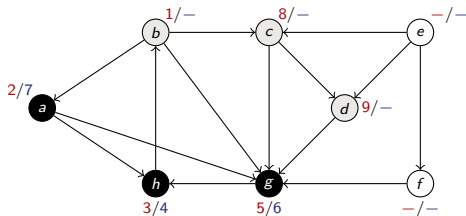
- ▶ WHITE = undiscovered
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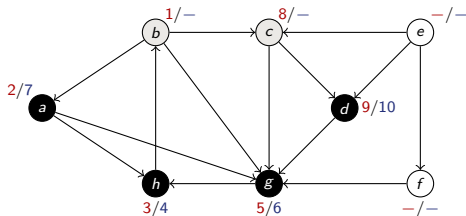


time = 9

Example of DFS

As DFS progresses, every vertex has a color:

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- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)

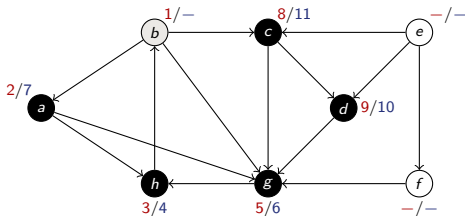


time = 10

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)

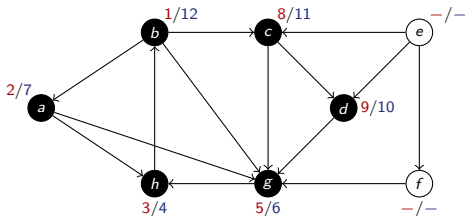


time = 11

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)

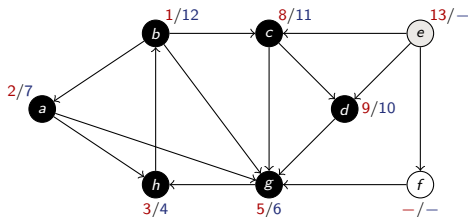


time = 12

Example of DFS

As DFS progresses, every vertex has a color:

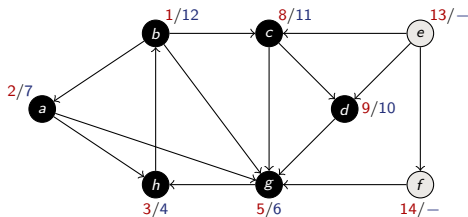
- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
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As DFS progresses, every vertex has a color:

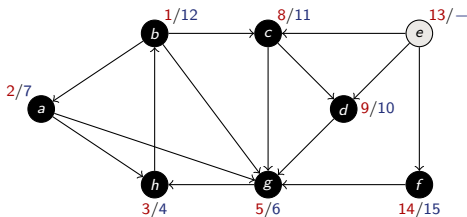
- ▶ WHITE = undiscovered
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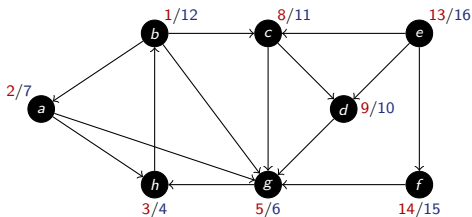


time = 15

Example of DFS

As DFS progresses, every vertex has a color:

- ▶ WHITE = undiscovered
- ▶ GRAY = discovered, but not finished (not done exploring from it)
- ▶ BLACK = finished (have found everything reachable from it)



Pseudocode of DFS

DFS(G)

for each $u \in G.V$

$u.color = \text{WHITE}$

$time = 0$

for each $u \in G.V$

if $u.color == \text{WHITE}$

 DFS-VISIT(G, u)

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

 // discover u

for each $v \in G.Adj[u]$

 // explore (u, v)

if $v.color == \text{WHITE}$

 DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

 // finish u

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

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// discover u

for each $v \in G.Adj[u]$

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if $v.color == \text{WHITE}$

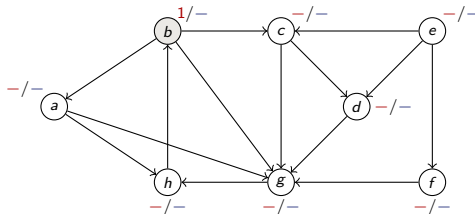
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



$time = 1$

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == \text{WHITE}$

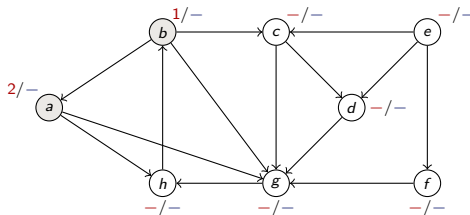
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$u.color = \text{BLACK}$

$time = time + 1$

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Pseudocode of DFS

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$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

for each $v \in G.Adj[u]$

// discover u

// explore (u, v)

if $v.color == \text{WHITE}$

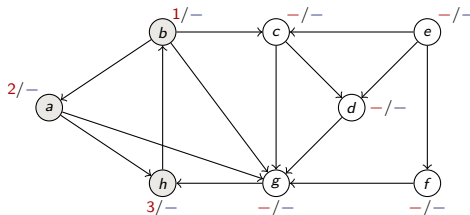
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



time = 3

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == \text{WHITE}$

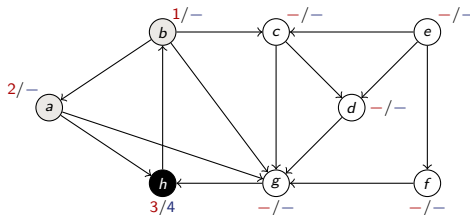
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



time = 4

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == \text{WHITE}$

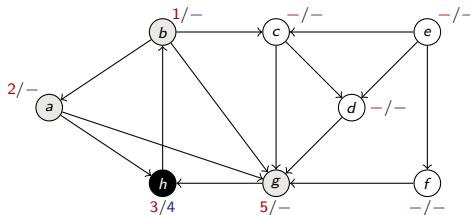
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



time = 5

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

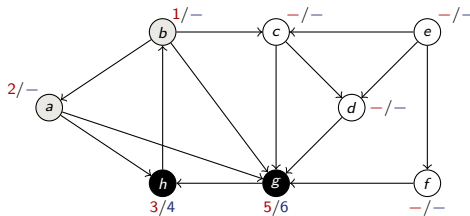
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

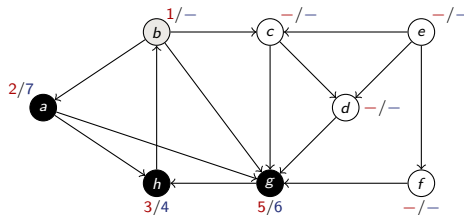
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 7

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == \text{WHITE}$

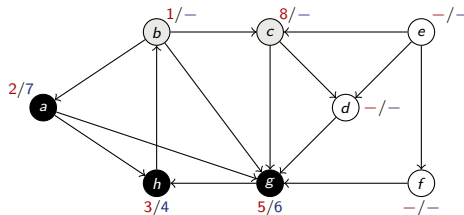
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



time = 8

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == \text{WHITE}$

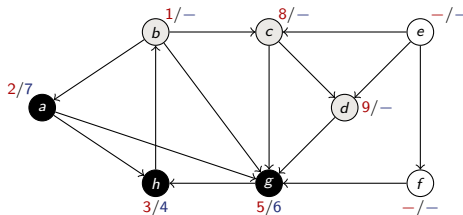
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



time = 9

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

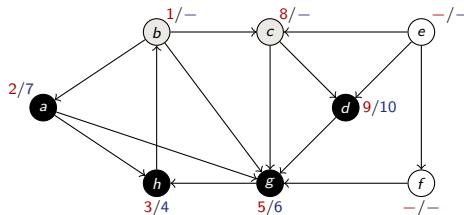
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 10

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

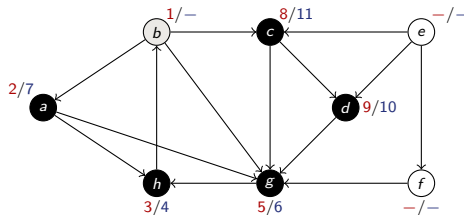
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 11

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

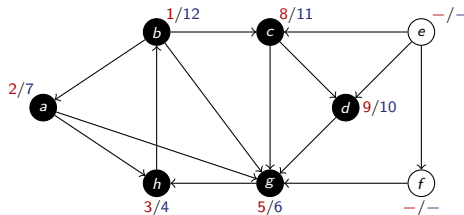
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 12

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

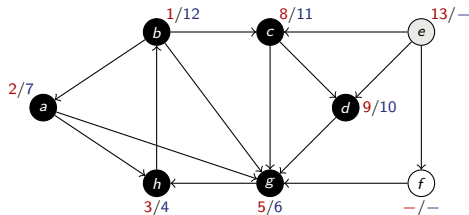
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 13

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

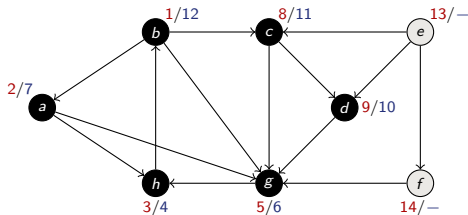
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 14

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = \text{GRAY}$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == \text{WHITE}$

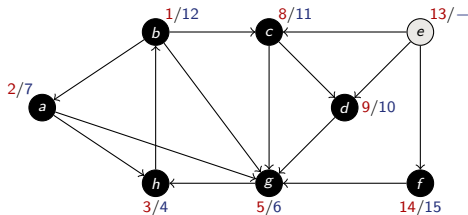
DFS-VISIT(v)

$u.color = \text{BLACK}$

$time = time + 1$

$u.f = time$

// finish u



time = 15

Pseudocode of DFS

DFS-VISIT(G, u)

$time = time + 1$

$u.d = time$

$u.color = GRAY$

// discover u

for each $v \in G.Adj[u]$

// explore (u, v)

if $v.color == WHITE$

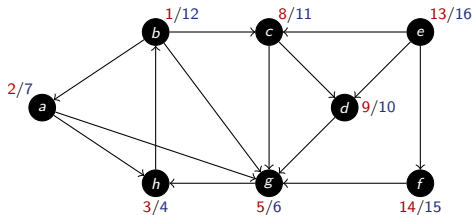
DFS-VISIT(v)

$u.color = BLACK$

$time = time + 1$

$u.f = time$

// finish u



time = 16

Analysis

DFS forms a **depth-first forest** comprised of ≥ 1 **depth-first trees**. Each tree is made of edges (u, v) such that u is gray and v is white when (u, v) is explored.

Analysis

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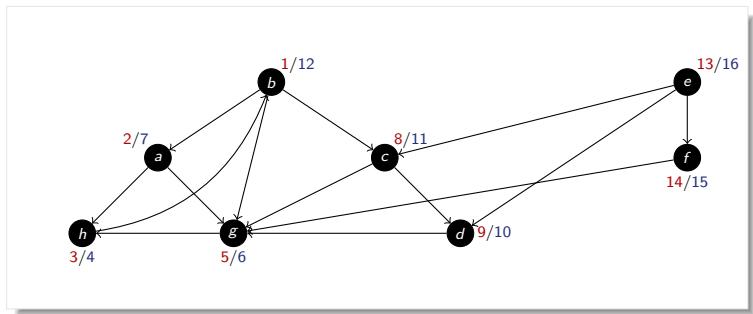
Runtime analysis:

DFS forms a **depth-first forest** comprised of ≥ 1 **depth-first trees**. Each tree is made of edges (u, v) such that u is gray and v is white when (u, v) is explored.

Runtime analysis: $\Theta(V + E)$

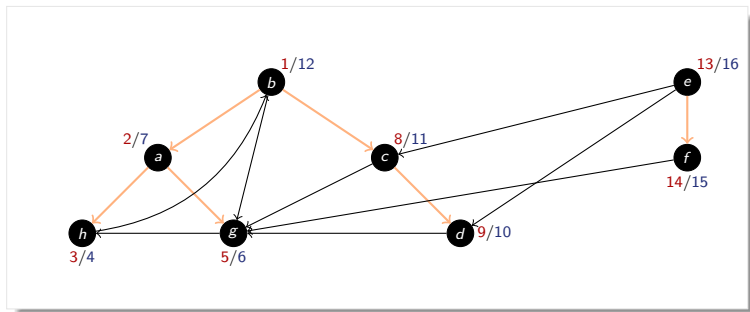
- ▶ $\Theta(V)$ because each vertex is discovered once
- ▶ $\Theta(E)$ because each edge is examined once if directed graph and twice if undirected graph.

Classification of edges



Classification of edges

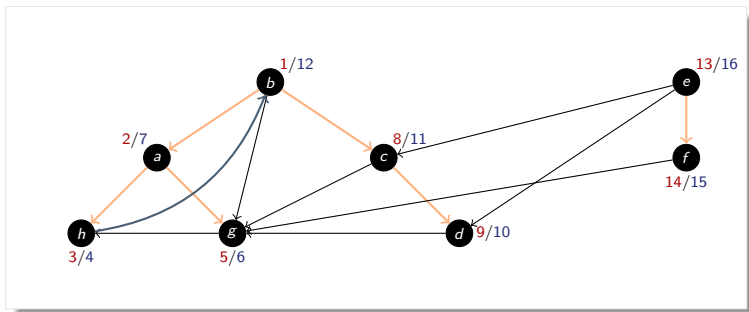
Tree edge: In the depth-first forest, found by exploring (u, v)



Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

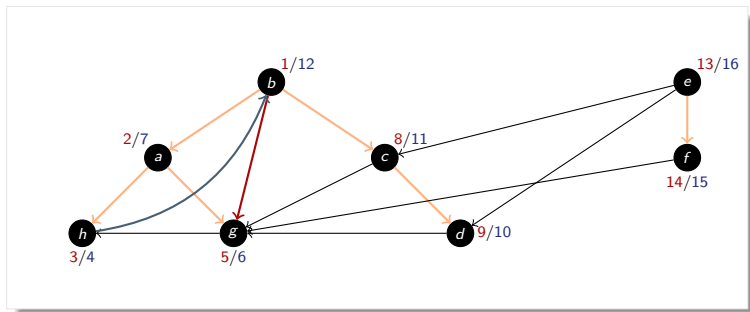


Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

Forward edge: (u, v) where v is a descendant of u , but not a tree edge



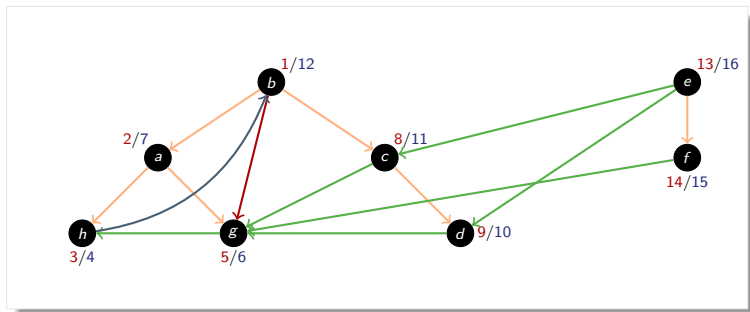
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Cross edge: any other edge



Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

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In DFS of an undirected graph we get only tree and back edges, no forward or cross-edges. Why?

